Journal of Algebra **237,** 406–419 (2001)
doi:10.1006/jabr.2000.8580, available online at http://www.idealibrary.com on IDEAL®

On the Set of Associated Primes of a Local Cohomology Module

M. Hellus

metadata, citation and similar papers at core.ac.uk

Communicated by Craig Huneke

Received May 19, 2000

Assume R is a local Cohen–Macaulay ring. It is shown that $\mathrm{Ass}_R(H_I^l(R))$ is finite for any ideal I and any integer l provided $\mathrm{Ass}_R(H^2_{(x,y)}(R))$ is finite for any x, $y \in R$ and $\mathrm{Ass}_R(H^3_{(x_1,x_2,y)}(R))$ is finite for any $y \in R$ and any regular sequence x_1, $x_2 \in R$. Furthermore it is shown that $\mathrm{Ass}_R(H_I^l(R))$ is always finite if $\dim(R) \leq 3$. The same statement is even true for $\dim(R) \leq 4$ if R is almost factorial. © 2001 Academic Press

Cohomology theory is an important part of algebraic geometry. If one considers local cohomology on an affine scheme with support in a closed subset, everything can be expressed in terms of rings, ideals, and modules. More precisely, let R be a noetherian ring and let I be an ideal of R (determining a closed subset of $\mathrm{Spec}(R)$): In this situation one studies the local cohomology modules $H_I^l(M)$, where l is a natural number and M is any R-module. As these local cohomology modules behave well under localization, one often restricts the above situation to the case R is a local ring.

As the structure of local cohomology modules in general seems to be quite mysterious, one tries to establish finiteness properties providing a better understanding of these modules. Finiteness properties of local cohomology modules have been studied by several authors; see for example Brodmann and Lashgari Faghani [1], Huneke and Koh [5], Huneke and Sharp [6], Lyubeznik [8], and Singh [11]. For a survey of results see Huneke [7].

Throughout this paper $(R, \mathfrak{m})$ is a local noetherian ring and I is an ideal of R. We deal with the question of whether the set of associated

0021-8693/01 $35.00
Copyright © 2001 by Academic Press
All rights of reproduction in any form reserved.

primes of every local cohomology module $H_I^l(R)$ is finite. As local cohomology modules in general are not finitely generated, this is an interesting question. For example if R is a regular local ring containing a field then $H_I^l(R)$ (for $l \geq 1$) is finitely generated only if it vanishes. This is true, because Lyubeznik [8, 9] proved

$$\operatorname{injdim}\big(H_I^l(R)\big) \leq \dim\big(\operatorname{Supp}_R\big(H_I^l(R)\big)\big)$$

for any ideal I and any l. Now if $0 \neq H_I^l(R)$ was finitely generated, we would have from [10, Theorem 18.9],

$$\dim(R) = \operatorname{depth}(R) = \operatorname{injdim}\big(H_I^l(R)\big) \leq \dim\big(\operatorname{Supp}_R\big(H_I^l(R)\big)\big) \leq \dim(R)$$

and consequently $\operatorname{Supp}_R(H_I^l(R)) = \operatorname{Spec}(R)$, contradicting $l \geq 1$.

In [3] Grothendieck conjectured that at least $\operatorname{Hom}_R(R/I, H_I^l(R))$ is always finitely generated, but soon Hartshorne was able to present the following counterexample to Grothendieck's conjecture (see [4] for details and a proof): Let k be a field, let $R = k[X,Y,Z,W]/(XY - ZW) = k[x,y,z,w]$, and let I be the ideal $(x,z) \subseteq R$. Then $\operatorname{Hom}_R(R/I, H_I^2(R))$ is not finitely generated.

However, in Hartshorne's example the ring R is not regular. Thus the question arises whether Grothendieck's conjecture is true at least in the regular case. In this context there is a theorem [5, Theorem 2.3(ii); 8, Corollary 3.5] stating that if I is an ideal of a regular ring R which contains a field and b is the maximum of the heights of all primes minimal over I then for $l > b$, $\operatorname{Hom}_R(R/I, H_I^l(R))$ is finitely generated if and only if $H_I^l(R) = 0$.

Using this theorem one can give a counterexample to Grothendieck's conjecture in the regular case, an idea which is due to Hochster:

Let k be a field of characteristic zero, let $R = k[[X_1, \dots, X_6]]$ be a power series ring in six variables, and let I_Δ be the ideal generated by the 2×2-minors of the matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}.$$

It can be seen that I_Δ has pure height two and that $H_{I_\Delta}^3(R)$ does not vanish. Now the above theorem implies $\operatorname{Hom}_R(R/I, H_{I_\Delta}^3(R))$ is not finitely generated. But Theorem 7(a) shows that at least the set of associated primes of $\operatorname{Hom}_R(R/I, H_{I_\Delta}^3(R))$ (which is the same as $\operatorname{Ass}_R(H_{I_\Delta}^3(R))$) is finite.

So one may wonder if any local cohomology module has only finitely many associated primes. In [7] Huneke conjectured the following: If R is a local noetherian ring, then $\mathrm{Ass}_R(H_I^l(R))$ is finite for any I and any l. This paper deals with a weaker version of Huneke's conjecture:

Conjecture $(*)$. If R is a local Cohen–Macaulay ring, then $\mathrm{Ass}_R(H_I^l(R))$ is finite for any I and any l.

Our main result is:

THEOREM 6. *If R is a local Cohen–Macaulay ring, the following are equivalent*:

(i) $(*)$ *is true for R.*

(ii) *The following two conditions are fulfilled*:

(a) $\mathrm{Ass}_R(H^2_{(x,y)}(R))$ *is finite for every $x, y \in R$.*

(b) $\mathrm{Ass}(H^3_{(x_1,x_2,y)}(R))$ *is finite, whenever x_1, $x_2 \in R$ is a regular sequence and $y \in R$.*

In Remark 2 it is shown that in the regular case condition (ii), (a) *is always satisfied. In fact at this point we will not assume that R is regular. We only need R to be a so-called almost factorial ring, which is weaker than being factorial.*

Besides this main result conjecture $(*)$ is proved in several special cases, for example in case $\dim(R) \leq 3$ or furthermore in case $\dim(R) \leq 4$ provided R is almost factorial.

Before going into the details, we remark that in the sequel we use a certain (first-quadrant cohomological) spectral-sequence, the so-called Groethendieck spectral-sequence for composed functors:

If I and J are ideals of a noetherian ring R, there is a converging spectral-sequence

$$E_2^{p,q} = H_I^p\big(H_J^q(M)\big) \Rightarrow H_{I+J}^{p+q}(M)$$

for every R-module M: This is true because Γ_J of an injective module is injective again, where $\Gamma_J(M)$ is defined as the submodule $\{m \in M \mid J^n \cdot m = 0 \text{ for some } n\}$ of M (for details see [12, Theorem 5.8.3).

We now start our examination of conjecture $(*)$: At least for the spot $l = \mathrm{depth}(I, R)$ there are only finitely many associated primes:

THEOREM 1. *Let $(R, \mathfrak{m})$ be a noetherian local ring, let M be a finitely generated R-module, and let $I \subseteq R$ be an ideal. Set $t = \mathrm{depth}(I, M)$. Then*

$$\mathrm{Ass}_R\big(H_I^t(M)\big) \subseteq \mathrm{Ass}_R\big(\mathrm{Ext}_R^t(R/I, M)\big)$$

and so $\mathrm{Ass}_R(H_I^t(M))$ is finite.

Proof. Choose $\mathfrak{p} \in \mathrm{Ass}_R(H_I^t(M))$ arbitrarily. Because of $H_{IR_\mathfrak{p}}^t(M_\mathfrak{p}) \neq 0$ we must have $t = \mathrm{depth}(IR_\mathfrak{p}, M_\mathfrak{p})$ and so we may assume $\mathfrak{p} = \mathfrak{m}$. Considering the structure of $H_I^t(M)$ as a direct limit of certain Ext-modules we conclude

$$\mathrm{Hom}_R\big(R/\mathfrak{m}, \mathrm{Ext}_R^t(R/I^n, M)\big) \neq 0$$

for some $n \in \mathbb{N}$. Let $x_1, \ldots, x_t \in I$ be a regular sequence. Using well-known formulas concerning Ext we get

$$\begin{aligned} 0 &\neq \mathrm{Hom}_R\big(R/\mathfrak{m}, \mathrm{Ext}_R^t(R/I^n, M)\big) \\ &= \mathrm{Hom}_R\big(R/\mathfrak{m}, \mathrm{Hom}_R(R/I^n, M/(x_1^n, \ldots, x_t^n)M)\big) \\ &= \mathrm{Hom}_R\big(R/\mathfrak{m}, \mathrm{Hom}_R(R/I, M/(x_1^n, \ldots, x_t^n)M)\big) \\ &= \mathrm{Hom}_R\big(R/\mathfrak{m}, \mathrm{Ext}_R^t(R/I, M)\big). \end{aligned}$$

Now it follows that $\mathfrak{m} \in \mathrm{Ass}_R(\mathrm{Ext}_R^t(R/I, M))$.

A theorem established by Brodmann and Lashgari Faghani [1, Proposition 2.1] states something more general: Let R be a noetherian ring, let $\mathfrak{a} \subseteq R$ be an ideal, and let M be a finitely generated R-module. Furthermore, let $i \in \mathbb{N}$ be given such that $H_\mathfrak{a}^j(M)$ is finitely generated for all $j < i$ and let $N \subseteq H_\mathfrak{a}^i(M)$ be a finitely generated submodule. Then, the set $\mathrm{Ass}_R(H_\mathfrak{a}^i(M)/N)$ is finite.

LEMMA 1. *Let R be a noetherian ring, let M be an R-module, and let I, J be ideals of R with $\sqrt{I} \subseteq \sqrt{J}$. Then*

$$H_I^l(M) = H_I^l(M/\Gamma_J(M))$$

for any $l \geq 1$.

Proof. Considering the long exact Γ_I-cohomology-sequence belonging to

$$0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0,$$

we see it suffices to show $H_I^l(\Gamma_J(M)) = 0$ for $l \geq 1$. Writing M as the union of its finitely generated submodules, we reduce to the case M itself is finitely generated, so that $\Gamma_J(M)$ is an R/J^n-module ($n \gg 0$). Consequently

$$H_I^l(\Gamma_J(M)) = H_{I(R/J^n)}^l(\Gamma_J(M)) = H_{(0)}^l(\Gamma_J(M)) = 0.$$

Theorem 1 treated the case $l = \mathrm{depth}(I, R)$, and our next theorem deals with the case $l = 1$:

THEOREM 2. *Let R be a noetherian local ring, let $I \subseteq R$ be an ideal, and let M be a finitely generated R-module. Then* $\operatorname{Ass}_R(H_I^1(M))$ *is contained in* $\operatorname{Ass}_R(\operatorname{Ext}_R^1(R/I, M/\Gamma_I(M)))$ *and hence is finite.*

Proof. From Lemma 1 we get

$$H_I^1(M) = H_I^1(M/\Gamma_I(M))$$

and $\Gamma_I(M/\Gamma_I(M)) = 0$ implies $\operatorname{depth}(I, M/\Gamma_I(M)) \geq 1$. So Theorem 2 becomes a corollary of Theorem 1.

The next theorem shows that in studying conjecture $(*)$, it suffices to examine $H_I^j(R)$ when $\operatorname{height}(I)$ equals $j-1$ or j.

THEOREM 3. *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring, let $I \subseteq R$ be an ideal, let $j >$* $\operatorname{height}(I)$, *and let $H_I^j(R) \neq 0$. Then there exists an ideal $\tilde{I} \supseteq I$ of height $j-1$ such that the natural homomorphism*

$$H_{\tilde{I}}^j(R) \to H_I^j(R)$$

becomes an isomorphism.

Proof. We may assume $\operatorname{height}(I) < j-1$. Set $t = \operatorname{height}(I)$ and let $x_1, \ldots, x_t \in I$ be a regular sequence. We denote the associated primes of $R/(x_1, \ldots, x_t)$ by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, enumerated in such a way that

$$I \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$
$$I \nsubseteq \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n.$$

We necessarily have $r < n$, because otherwise $\sqrt{I} = \sqrt{(x_1, \ldots, x_t)}$ and consequently $H_I^j(R) = 0$, contrary to the assumptions. Using prime avoidance we choose

$$y \in (\mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_n) \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$$

and consider the Mayer–Vietoris sequence with respect to the ideals (y), I and the R-module $H_{(x_1, \ldots, x_t)}^t(R) =: M$:

$$H_{I \cap (y)}^{j-t-1}(M) \to H_{(I,y)}^{j-t}(M) \to H_I^{j-t}(M) \oplus H_{(y)}^{j-t}(M)$$
$$\to H_{I \cap (y)}^{j-t}(M).$$

In the sequel we write $(\underline{x})$ for the ideal $(x_1, \ldots, x_t)$ of R. Because $j - t \geq 2$ and $I \cap (y) \subseteq \sqrt{(\underline{x})}$ it follows that $H_{(y)}^{j-t} = 0$ and both the leftmost and rightmost term in this sequence vanish; so the second arrow is an isomorphism. Using the spectral-sequences for the composed functors $\Gamma_{(I,y)} \circ \Gamma_{(\underline{x})}$ and $\Gamma_I \circ \Gamma_{(\underline{x})}$ we conclude

$$\begin{aligned} H^j_{(I,y)}(R) &= H^{j-t}_{(I,y)}(M) \\ &= H^{j-t}_I(M) \\ &= H^j_I(R). \end{aligned}$$

By construction height(I, y) = height$(I) + 1$. Now the statement of the theorem follows inductively.

The following corollary is the first step in a series of reductions of conjecture $(*)$:

COROLLARY 1. *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring and let $j \in \mathbb{N}$. Then the following two statements are equivalent*:

(i) $\mathrm{Ass}_R(H^j_I(R))$ *is finite for each ideal* $I \subseteq R$.

(ii) $\mathrm{Ass}_R(H^j_I(R))$ *is finite for each ideal* $I \subseteq R$ *satisfying* height$(I) = j - 1$.

Proof. This follows immediately from Theorem 3.

Using the ideas of the proof of Theorem 3 one can show that $H^j_I(R)$ has only finitely many associated primes of height j:

COROLLARY 2. *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring, let I be an ideal of R, and let $j \in \mathbb{N}$. Then*

$$\mathrm{Supp}_R\big(H^j_I(R)\big) \cap \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{height}(\mathfrak{p}) = j\}$$

is finite and therefore $H^j_I(R)$ has only finitely many associated prime ideals of height j.

Proof. We may assume height$(I) \leq j - 1$. Because of Theorem 3 we may even assume that the height of I equals $j - 1$. Let $x_1, \dots, x_{j-1} \in I$ be a regular sequence and let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the associated primes of $R/(x_1, \dots, x_{j-1})$, enumerated in such a way that we have

$$\begin{gathered} I \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r, \\ I \not\subseteq \mathfrak{p}_{r+1}, \dots, \mathfrak{p}_n. \end{gathered}$$

We assume $r < n$ (if $r = n$ we have $\sqrt{I} = \sqrt{(\underline{x})}$ and therefore $H^j_I(R) = 0$). Set $J := \mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_n$ and consider the following part of a Mayer–Vietoris sequence:

$$H^j_{I+J}(R) \to H^j_I(R) \oplus H^j_J(R) \to H^j_{(x_1, \dots, x_{j-1})}(R) = 0.$$

It follows that $\operatorname{Supp}_R(H_I^j(R)) \subseteq \mathscr{V}(I+J)$. As $\operatorname{height}(I+J) \geq j$, there are only finitely many primes of height j in $\operatorname{Supp}_R(H_I^j(R))$.

The methods we have developed so far suffice to prove conjecture $(*)$ in the case $\dim(R) \leq 3$:

COROLLARY 3. *Let R be a local Cohen–Macaulay ring of dimension at most three, let I be an ideal of R, and let $j \in \mathbb{N}$. Then $H_I^j(R)$ has only finitely many associated primes.*

Proof.

Case $\dim(R) = 2$. If $j = 2$, the statement follows immediately from Theorems 1 and 3. The case $j = 1$ is done by Theorem 2.

Case $\dim(R) = 3$. The case $j = 3$ follows at once from Theorems 1 and 3. $j = 1$ is again done by Theorem 2. If $j = 2$, we assume $\operatorname{height}(I) = 1$ by Theorem 2. Now the statement follows from Corollary 2.

LEMMA 2. *Let I be an ideal of a noetherian ring R and let M be any R-module. Then* $\operatorname{Ass}_R(M/\Gamma_I(M)) = \operatorname{Ass}_R(M) \cap (\operatorname{Spec}(R) \setminus \mathscr{V}(I))$.

Proof. If $\mathfrak{p}$ is associated to $M/\Gamma_I(M)$ we get from an exact sequence

$$0 \to R/\mathfrak{p} \to M/\Gamma_I(M)$$

an exact sequence

$$0 \to \Gamma_I(R/\mathfrak{p}) \to \Gamma_I(M/\Gamma_I(M)) = 0$$

and consequently $\mathfrak{p}$ does not contain I. Choose $m \in M$ satisfying $\Gamma_I(M) : m = \mathfrak{p}$. Localizing we conclude

$$0 : \frac{m}{1} = \Gamma_{IR_\mathfrak{p}}(M_\mathfrak{p}) : \frac{m}{1} = \mathfrak{p} R_\mathfrak{p}.$$

From our assumptions it follows that $\frac{m}{1} \neq 0$, because otherwise there would exist $s \in R \setminus \mathfrak{p}$ with $sm = 0$, contradicting $\Gamma_I(M) : m = \mathfrak{p}$. Hence $\mathfrak{p} R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$; equivalently $\mathfrak{p} \in \operatorname{Ass}_R(M)$.

On the other hand, if we choose $\mathfrak{p} \in \operatorname{Ass}_R(M) \cap (\operatorname{Spec}(R) \setminus \mathscr{V}(I))$, $\mathfrak{p}$ cannot be associated to $\Gamma_I(M)$ and consequently must be associated to $M/\Gamma_I(M)$ (consider $0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$ exact).

LEMMA 3. *Let I be an ideal of a local Cohen–Macaulay ring R and set* $l = \operatorname{height}(I) + 1$. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the elements of* $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p}$ *minimal over I and* $\operatorname{height}(\mathfrak{p}) = \operatorname{height}(I)\}$. *Set $I^{\text{pure}} := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. Then finiteness of* $\operatorname{Ass}_R(H^l_{I^{\text{pure}}}(R))$ *implies finiteness of* $\operatorname{Ass}_R(H_I^l(R))$.

Proof. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the elements of $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p}$ minimal over I and $\operatorname{height}(\mathfrak{p}) > \operatorname{height}(I)\}$ (without restriction assume $m \geq 1$) and set $I'' := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$. Then $\sqrt{I} = I^{\text{pure}} \cap I''$. Consider the Mayer–Vietoris sequence

$$H^l_{I^{\text{pure}} + I''}(R) \to H^l_{I^{\text{pure}}}(R) \oplus H^l_{I''}(R) \to H^l_I(R) \to H^{l+1}_{I^{\text{pure}} + I''}(R).$$

As by construction $\operatorname{height}(I^{\text{pure}} + I'') \geq \operatorname{height}(I) + 2 = l + 1$, the leftmost term in this sequence vanishes and the rightmost term has only finitely many associated primes. Furthermore $\operatorname{height}(I'') \geq \operatorname{height}(I) + 1 = l$ and so $H^l_{I''}(R)$ has only finitely many associated prime ideals. Now the statement of the lemma is obvious.

Now we are in a position to give the next reduction of conjecture $(*)$, which roughly spoken says one may restrict to the case $j = \mu(I)$ when examining $\operatorname{Ass}_R(H^j_I(R))$:

THEOREM 4. *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring and let $t \in \mathbb{N}$. Then the following two statements are equivalent*:

(i) *$H^{t+1}_I(R)$ has only finitely many associated prime ideals for each ideal I of R.*

(ii) *Whenever $x_1, \ldots, x_t \in R$ is a regular sequence and $y \in R$, the module $H^{t+1}_{(x_1, \ldots, x_t, y)}(R)$ has only finitely many associated prime ideals.*

Proof. Assume condition (ii) is satisfied and let I be an arbitrary ideal of R. We have to show $\operatorname{Ass}_R(H^{t+1}_I(R))$ is finite. Using Corollary 1 we may assume $\operatorname{height}(I) = t$. Using Lemma 3 we can even assume that all primes minimal over I have height t.

Let $x_1, \ldots, x_t \in I$ be a regular sequence and denote the primes minimal over I by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. These are also minimal over $(x_1, \ldots, x_t)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the other primes minimal over $(x_1, \ldots, x_t)$ (that is, the ones not containing I). As all the ideals $\mathfrak{p}_i$ and $\mathfrak{q}_j$ have height t, we may choose a

$$y' \in (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n) \setminus (\mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_m).$$

Now a suitable power y of y' will satisfy

$$y \in I \setminus (\mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_m).$$

By using Lemma 2 it follows that y is not in any prime ideal associated to the R-module $(R/(x_1^s, \ldots, x_t^s))/\Gamma_I(R/(x_1^s, \ldots, x_t^s))$ ($s \in \mathbb{N}$ arbitrary). Consequently y operates injectively on $(R/(x_1^s, \ldots, x_t^s))/\Gamma_I(R/(x_1^s, \ldots, x_t^s))$.

From the exactness of the direct limit-functor we conclude that y operates injectively on

$$\begin{aligned}
&\lim_{\substack{\rightarrow\\ s\in N}}\left[\left(R/(x_1^s,\dots,x_t^s)\right)/\Gamma_I\left(R/(x_1^s,\dots,x_t^s)\right)\right]\\
&= \lim_{\substack{\rightarrow\\ s\in N}}\left(R/(x_1^s,\dots,x_t^s)\right)/\Gamma_I\left(\lim_{\substack{\rightarrow\\ s\in N}}\left(R/(x_1^s,\dots,x_t^s)\right)\right)\\
&= H^t_{(x_1,\dots,x_t)}(R)/\Gamma_I\left(H^t_{(x_1,\dots,x_t)}(R)\right).
\end{aligned}$$

Call this property of y $(**)$. From well-known spectral-sequence arguments it follows that

$$\begin{aligned}
H_I^{t+1}(R) &= H_I^1\left(H^t_{(x_1,\dots,x_t)}(R)\right)\\
&\overset{(+)}{=} H_I^1\left(H^t_{(x_1,\dots,x_t)}(R)/\Gamma_I\left(H^t_{(x_1,\dots,x_t)}(R)\right)\right)\\
&\overset{(**)}{=} \Gamma_I\left(H^1_{(y)}\left(H^t_{(x_1,\dots,x_t)}(R)/\Gamma_I\left(H^t_{(x_1,\dots,x_t)}(R)\right)\right)\right)\\
&\subseteq H^1_{(y)}\left(H^t_{(x_1,\dots,x_t)}(R)/\Gamma_I\left(H^t_{(x_1,\dots,x_t)}(R)\right)\right)\\
&\overset{(+)}{=} H^1_{(y)}\left(H^t_{(x_1,\dots,x_t)}(R)\right)\\
&= H^{t+1}_{(x_1,\dots,x_t,y)}(R).
\end{aligned}$$

The two equalities $(+)$ follow from Lemma 1. The above inclusion finishes our proof, since we can conclude

$$|\mathrm{Ass}_R\left(H_I^{t+1}(R)\right)| \le |H^{t+1}_{(x_1,\dots,x_t,y)}(R)| < \infty.$$

Using the various statements established so far, we can prove conjecture $(*)$ in the case R is regular of dimension at most four (cf. Theorem 5); in fact we do not actually need R to be regular. We will only use the fact that every height one prime ideal is principal up to radical; this is true if R is a Krull domain whose divisor class group is torsion (cf. [2, Proposition 6.8]). Krull domains whose divisor class group is torsion are usually called almost factorial. In particular if R is factorial, it is almost factorial.

THEOREM 5. *Let R be a local almost factorial Cohen–Macaulay ring of dimension at most four, let I be an ideal of R, and let $j \in \mathbb{N}$. Then $H_I^j(R)$ has only finitely many associated primes; that is, in these cases conjecture $(*)$ is true.*

Proof. We may restrict ourselves to the case $\dim(R) = 4$. The case $j = 0$ is trivial, $j = 1$ follows from Theorem 2, $j = 3$ follows from our

Corollaries 1 and 2, and $j = 4$ follows from Theorem 3. In the remaining case $j = 2$ we may assume height$(I) = 1$ (Theorem 3). Using Lemma 3, we may even assume that all primes minimal over I have height one. In our case this means that I is principal up to radical and so $H_I^2(R) = 0$.

Theorem 6 is our final reduction of conjecture $(*)$, allowing us to restrict ourselves to the examination of "two" special cases (for the regular case, see Remark 2):

THEOREM 6. *Let R be a local Cohen–Macaulay ring. Then the following two statements are equivalent*:

(i) *$H_I^j(R)$ has only finitely many associated prime ideals for each ideal I of R and each $j \in \mathbb{N}$.*

(ii) *The following two conditions are satisfied*:

(a) *$\mathrm{Ass}_R(H^2_{(x,y)}(R))$ is finite for every $x, y \in R$.*

(b) *$\mathrm{Ass}_R(H^3_{(x_1,x_2,y)}(R))$ is finite whenever $x_1, x_2 \in R$ is a regular sequence and $y \in R$.*

Proof. We only have to show (ii) implies (i): We do this by induction on j:

- $j = 0$: Easy.
- $j = 1$: Theorem 2.
- $j = 2, 3$: Theorem 4.
- $j \geq 4$: Using Theorem 4 we assume that $I = (x_1, \dots, x_j)$ (for some $x_1, \dots, x_j \in R$). Here [ ] means Gaussian brackets; that is, $[q] := \max\{i \in \mathbf{Z} \mid i \leq q\}$ for rational q. Set $I' := (x_1, \dots, x_{[j/2]})$, $I'' := (x_{[j/2]+1}, \dots, x_j) \subseteq R$ ideals and consider the following Mayer–Vietoris sequence:

$$H_{I'}^{j-1}(R) \oplus H_{I''}^{j-1}(R) \to H_{I' \cap I''}^{j-1}(R) \to H_I^j(R) \to H_{I'}^j(R) \oplus H_{I''}^j(R).$$

Combined with our induction hypothesis (using $j - 1 \geq j - ([j/2] + 1) + 1$) we get from this an isomorphism

$$H_{I' \cap I''}^{j-1}(R) \to H_I^j(R).$$

Another application of our induction hypothesis finishes the proof of the theorem.

Remark 1. (i) Let R be a local Cohen–Macaulay ring, let $n \in \{2, 3\}$, and let $x_1, \dots, x_n \in R$. Now from $|\mathrm{Ass}_R(H^n_{(x_1,\dots,x_n)}(R))| < \infty$ conjecture $(*)$ would follow. We can write the module $H^n_{(x_1,\dots,x_n)}(R)$ in another way. First we have

$$H^n_{(x_1,\dots,x_n)}(R) = H^1_{(x_1)}\Big(H^{n-1}_{(x_2,\dots,x_n)}(R)\Big)$$

and from the right-exactness of $H^1_{(x_1)}$ we may conclude

$$H^1_{(x_1)}\big(H^{n-1}_{(x_2,\dots,x_n)}(R)\big) = H^1_{(x_1)}(R) \otimes_R H^{n-1}_{(x_2,\dots,x_n)}(R).$$

An easy induction proof gives us

$$\begin{aligned} H^n_{(x_1,\dots,x_n)}(R) &= H^1_{(x_1)}(R) \otimes_R \cdots \otimes_R H^1_{(x_n)}(R) \\ &= \big(R_{x_1}/R\big) \otimes_R \cdots \otimes_R \big(R_{x_n}/R\big). \end{aligned}$$

So for conjecture $(*)$ it is sufficient to prove

$$|\mathrm{Ass}_R\big(\big(R_{x_1}/R\big) \otimes_R \cdots \otimes_R \big(R_{x_n}/R\big)\big)| < \infty$$

for $n \in \{2,3\}$.

(ii) Consider the complete case; that is, R is a local complete Cohen–Macaulay ring. Similar to Theorem 6, condition (ii) assume $t \in \{1,2\}$, $x_1,\dots,x_t \in R$ a regular sequence, and $y \in R$. Consider R as an $R[[T]]$-module via the R-algebra homomorphism $R[[T]] \to R$ sending T to y. We then calculate

$$\begin{aligned} H^{t+1}_{(x_1,\dots,x_t,y)}(R) &= H^{t+1}_{(x_1,\dots,x_t,T)}(R) \\ &= H^{t+1}_{(x_1,\dots,x_t,T)}(R[[T]]/(T-y)) \\ &= H^{t+1}_{(x_1,\dots,x_t,T)}(R[[T]])/(T-y)H^{t+1}_{(x_1,\dots,x_t,T)}(R[[T]]). \end{aligned}$$

Since $x_1,\dots,x_t$, $T \in R[[T]]$ is a regular sequence, it is in the complete case sufficient (for conjecture $(*)$) to show that whenever $t \in \{2,3\}$, $x_1,\dots,x_t \in R$ is a regular sequence and whenever $y \in R$ we have

$$|\mathrm{Ass}_R\big(H^t_{(x_1,\dots,x_t)}(R)/yH^t_{(x_1,\dots,x_t)}(R)\big)| < \infty.$$

Remark 2. If R is an almost factorial local ring, condition (a) from Theorem 6(ii) is automatically fulfilled. To show this we may, with respect to Theorem 3, assume $\mathrm{height}(x,y) = 1$. Using Lemma 3 we may even assume that all primes minimal over (x,y) have height one. As R is almost factorial, it follows that (x,y) is principal up to radical and so $H^2_{(x,y)}(R) = 0$.

The remaining Theorems 7 and 8 prove conjecture $(*)$ in certain generic cases (where R/I is Cohen–Macaulay); Theorem 7 treats the equicharacteristic case and Theorem 8 deals with mixed characteristics.

THEOREM 7. (a) *Let k be a field, let $R = k[[X_1, \ldots, X_6]]$ be a power series ring in six indeterminates, let $\Delta_1 := X_2X_6 - X_3X_5$, $\Delta_2 := X_1X_6 - X_3X_4$, and $\Delta_3 := X_1X_5 - X_2X_4$ (these are the 2×2-minors of the matrix*

$$\left(\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}\right),$$

and let I be the ideal $(\Delta_1, \Delta_2, \Delta_3) \subseteq R$. Then $\operatorname{Supp}_R(H_I^3(R)) \subseteq \{(X_1, \ldots, X_6)\}$ and consequently $\operatorname{Ass}_R(H_I^3(R))$ is finite.

(b) *Let R be a local equicharacteristic Cohen–Macaulay ring and let $x_1, \ldots, x_6 \in R$ be a regular sequence. Let $\delta_1 := x_2x_6 - x_3x_5$, $\delta_2 := x_1x_6 - x_3x_4$, $\delta_3 := x_1x_5 - x_2x_4$, and let I be the ideal $(\delta_1, \delta_2, \delta_3) \subseteq R$. Then $\operatorname{Ass}_R(H_I^3(R))$ is finite.*

Proof. (a) It is well known that R/I is a Cohen–Macaulay domain of dimension 4. Consequently I is a prime ideal of height two. From [10, Theorem 30.4(ii)] it follows that

$$\operatorname{Sing}(R/(\Delta_1)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R/(\Delta_1)) \mid \mathfrak{p} \supseteq (X_2, X_6, X_3, X_5)\}.$$

Here $\operatorname{Sing}(R/(\Delta_1))$ means the set of all primes $\mathfrak{p}$ satisfying $(R/(\Delta_1))_{\mathfrak{p}}$ is not regular. Furthermore we have

$$\operatorname{Sing}(R/(\Delta_2)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R/(\Delta_1)) \mid \mathfrak{p} \supseteq (X_1, X_6, X_3, X_4)\}$$

and

$$\operatorname{Sing}(R/(\Delta_3)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R/(\Delta_1)) \mid \mathfrak{p} \supseteq (X_1, X_5, X_2, X_4)\}.$$

Choose $\mathfrak{p} \in \operatorname{Spec}(R/I) \setminus \{(X_1, \ldots, X_6)\}$ arbitrarily. We have to show $H^3_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$. From our above calculations we know there is an $i \in \{1, 2, 3\}$ with $\mathfrak{p} \notin \operatorname{Sing}(R/(\Delta_i))$. Thus $(R/(\Delta_i))_{\mathfrak{p}}$ is factorial. Combining this with the fact that $I/(\Delta_i)$ is a prime ideal of height one, we conclude the ideal $IR_{\mathfrak{p}}/(\Delta_i)R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}/(\Delta_i)R_{\mathfrak{p}}$ is principal. This finally shows

$$0 = H^2_{IR_{\mathfrak{p}}/(\Delta_i)R_{\mathfrak{p}}}\left(H^1_{(\Delta_i)R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right) = H^3_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}).$$

(b) We may assume that R is complete, because if the statement is proved in the complete case, then the formula

$$\operatorname{Ass}_R\left(H_I^3(R)\right) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(H_I^3 R(R))} \operatorname{Ass}_{\hat{R}}\left(\hat{R}/\mathfrak{p}\hat{R}\right)$$

(cf. [10, Theorem 23.2(ii)]) implies finiteness of $\operatorname{Ass}_R(H_I^3(R))$ (each $\operatorname{Ass}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$ contains a $\mathfrak{q}$ with $\mathfrak{q} \cap R = \mathfrak{p}$).

Let $k \subseteq R$ be a field, let $k[[X_1, \dots, X_6]]$ be a power series ring in six variables and let $\Delta_1, \Delta_2, \Delta_3 \in k[[X_1, \dots, X_6]]$ (like in (a)) be the 2×2-minors of

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}.$$

The flat k-algebrahomomorphism

$$k[[X_1, \dots, X_6]] \to R$$

with $X_i \mapsto x_i$ $(i = 1, \dots, 6)$ sends Δ_j to δ_j $(j = 1, 2, 3)$. This implies

$$H_I^3(R) = H^3_{(\Delta_1, \Delta_2, \Delta_3)}(R) = H^3_{(\Delta_1, \Delta_2, \Delta_3)}\big(k[[X_1, \dots, X_6]]\big) \otimes_{k[[X_1, \dots, X_6]]} R$$

and we conclude

$$\mathrm{Ass}_R\big(H_I^3(R)\big) \subseteq \mathrm{Ass}_R\big(R/(X_1, \dots, X_6)R\big),$$

from [10, Theorem 23.2(ii)], which finally proves (b).

THEOREM 8. (a) *Let p be a prime number, let C be a complete p-ring, let $R = C[[X_1, \dots, X_6]]$ be a power series ring in six variables, and set $\Delta_1 := X_2X_6 - X_3X_5$, $\Delta_2 := X_1X_6 - X_3X_4$, $\Delta_3 := X_1X_5 - X_2X_4$ (these are the 2×2-minors of the matrix*

$$\left(\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}\right),$$

and let I be the ideal $(\Delta_1, \Delta_2, \Delta_3) \subseteq R$. Then $\mathrm{Supp}_R(H_I^3(R)) \subseteq \mathscr{V}((X_1, \dots, X_6))$ and consequently $\mathrm{Ass}_R(H_I^3(R))$ is finite.

(b) *Let p be a prime number, let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring satisfying $\mathrm{char}(R) = 0$, $\mathrm{char}(R/\mathfrak{m}) = p$, and $x_1, \dots, x_6 \in R$ with the property that $p, x_1, \dots, x_6 \in R$ is a regular sequence. Set $\delta_1 := x_2x_6 - x_3x_5$, $\delta_2 := x_1x_6 - x_3x_4$, $\delta_3 := x_1x_5 - x_2x_4$ and let I be the ideal $(\delta_1, \delta_2, \delta_3) \subseteq R$. Then $\mathrm{Ass}_R(H_I^3(R))$ is finite.*

Proof. (a) The proof is practically the same as the proof of Theorem 7(a).

(b) Like in the proof of Theorem 7(b), we may assume that R is complete. According to [10, Theorem 29.3] R has a coefficient ring $C \subseteq R$.

Let $C[[X_1, \dots, X_6]]$ be a power series ring in six variables and let $\Delta_1, \Delta_2, \Delta_3 \in C[[X_1, \dots, X_6]]$ (like in (a)) be the 2×2-minors of

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}.$$

The rest of the proof may be copied from the proof of Theorem 7(b) until one finally gets

$$\operatorname{Ass}_R\big(H_I^3(R)\big) \subseteq \operatorname{Ass}_R\big(R/(X_1, \dots, X_6)R\big) \cup \operatorname{Ass}_R\big(R/(p, X_1, \dots, X_6)R\big),$$

which proves (b).

REFERENCES

1. M. P. Brodmann and A. Lashgari Faghani, A finiteness result for associated primes of local cohomology modules, preprint, 1998.
2. R. M. Fossum, "The Divisor Class Group of a Krull Domain," Springer-Verlag, Berlin/New York, 1973.
3. A. Grothendieck, Cohomologie locale des faiscaux cohérents et théorèmes de Lefschetz locaux et globaux, *S.G.A.* **2** (1968).
4. R. Hartshorne, Affine duality and cofiniteness, *Invent. Math.* **9** (1970), 145–164.
5. C. Huneke and J. Koh, Cofiniteness and vanishing of local cohomology modules, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), 421–429.
6. C. Huneke and R. Sharp, Bass numbers of local cohomology modules, *Trans. Amer. Math. Soc.* **339** (1993).
7. C. Huneke, Problems on local cohomology, *Res. Notes Math.* **2** (1992).
8. G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), *Invent. Math.* **113** (1993).
9. G. Lyubeznik, F-modules: Applications to local cohomology and D-modules in characteristic $p > 0$, preprint.
10. H. Matsumura, "Commutative Ring Theory," Cambridge Univ. Press, Cambridge, UK, 1986.
11. A. Singh, p-torsion elements in local cohomology modules, preprint, 1999.
12. C. A. Weibel, "An Introduction to Homological Algebra," Cambridge Univ. Press, Cambridge, UK, 1994.